Data-driven balancing for LQO systems

Reetish Padhi[1], Ion Victor Gosea[2], Igor Pontes Duff[2], Serkan Gugercin[1]

[1]Department of Mathematics, Virginia Tech, Blacksburg, VA 24061, USA
[2]Max Planck Institute for Dynamics of Complex Technical Systems, Magdeburg, Germany

1 Introduction

Linear dynamical systems with quadratic output (LQO) are characterized by:

$$\Sigma_{\mathrm{LQO}}:\begin{cases}\dot{\mathbf{x}}(t)=\mathbf{A}\mathbf{x}(t)+\mathbf{B}\mathbf{u}(t), & \mathbf{A}\in\mathbb{R}^{n\times n},\ \mathbf{B}\in\mathbb{R}^{n},\ \mathbf{u}\in\mathbb{R},\ \mathbf{x}\in\mathbb{R}^{n}\\ \mathbf{y}(t)=\mathbf{C}\mathbf{x}(t)+\mathbf{x}(t)^\top\mathbf{M}\mathbf{x}(t), & \mathbf{M}\in\mathbb{R}^{n\times n},\ \mathbf{C}^\top\in\mathbb{R}^{n},\ \mathbf{y}\in\mathbb{R}\end{cases}$$

The state equation is linear in $\mathbf{x}(t)$, while the output $\mathbf{y}(t)$ is a quadratic form in $\mathbf{x}(t)$.

The time-domain generalized kernel functions of system Σ_{LQO} are defined as follows,

$$\mathbf{h}_1(\tau_1)=\mathbf{C}e^{\mathbf{A}\tau_1}\mathbf{B},\quad \mathbf{h}_2(\tau_1,\tau_2)=\mathbf{B}^\top e^{\mathbf{A}^\top\tau_1}\mathbf{M}e^{\mathbf{A}\tau_2}\mathbf{B}.$$

The kernel $\mathbf{h}_1(t)$ describes the entire system in the linear time-invariant (LTI) case.

Goal: Construct a reduced model using balanced truncation (BT) from data (kernel samples)

$$\Sigma_r:\begin{cases}\dot{\mathbf{x}}_r(t)=\mathbf{A}_r\mathbf{x}_r(t)+\mathbf{B}_r\mathbf{u}(t), & \mathbf{A}_r\in\mathbb{R}^{r\times r},\ \mathbf{B}_r\in\mathbb{R}^{r},\ \mathbf{u}\in\mathbb{R},\ \mathbf{x}_r\in\mathbb{R}^{r}\\ \mathbf{y}_r(t)=\mathbf{C}_r\mathbf{x}_r(t)+\mathbf{x}_r(t)^\top\mathbf{M}_r\mathbf{x}_r(t), & \mathbf{M}_r\in\mathbb{R}^{r\times r},\ \mathbf{C}_r^\top\in\mathbb{R}^{r},\ \mathbf{y}\in\mathbb{R}\end{cases}$$

The observability Gramian $\mathbf{Q}\in\mathbb{R}^{n\times n}$ of system Σ_{LQO} is defined as $\mathbf{Q}=\mathbf{Q}_1+\mathbf{Q}_2$ as in [1]:

$$\mathbf{Q}_1=\int_0^\infty e^{\mathbf{A}^\top\tau}\mathbf{C}^\top\mathbf{C}e^{\mathbf{A}\tau}d\tau=\mathbf{L}_1\mathbf{L}_1^\top,\quad \mathbf{Q}_2=\int_0^\infty e^{\mathbf{A}^\top t}\mathbf{M}^\top\mathbf{P}\mathbf{M}e^{\mathbf{A}t}dt=\mathbf{L}_2\mathbf{L}_2^\top,$$

Let $\mathbf{L}$ be a square root factor of Gramian $\mathbf{Q}$ such that $\mathbf{Q}=\mathbf{L}\mathbf{L}^\top$. Additionally, the controllability Gramian $\mathbf{P}\in\mathbb{R}^{n\times n}$ is the same as in the standard LTI case, i.e.,

$$\mathbf{P}=\int_0^\infty e^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}^\top e^{\mathbf{A}^\top\tau}d\tau=\mathbf{U}\mathbf{U}^\top. \tag{1}$$

2 BT for LQO systems[1]

- **Input**: System matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{M}$.
- Compute the truncated **SVD** of matrix $\mathbf{L}^T\mathbf{U}$:

$$\mathbf{L}^T\mathbf{U}=\begin{bmatrix}\mathbf{Z}_1 & \mathbf{Z}_2\end{bmatrix}\begin{bmatrix}\mathbf{S}_1 & \\ & \mathbf{S}_2\end{bmatrix}\begin{bmatrix}\mathbf{Y}_1^\top\\ \mathbf{Y}_2^\top\end{bmatrix}.$$

- Construct $\mathbf{W}_r=\mathbf{L}\mathbf{Z}_1\mathbf{S}_1^{-1/2}$ and $\mathbf{V}_r=\mathbf{U}\mathbf{Y}_1\mathbf{S}_1^{-1/2}$.
- The matrices of the reduced-order balanced system are given by,

$$\mathbf{A}_r=\mathbf{W}_r^T\mathbf{A}\mathbf{V}_r=\mathbf{S}_1^{-1/2}\mathbf{Z}_1^T\left(\mathbf{L}^T\mathbf{A}\mathbf{U}\right)\mathbf{Y}_1\mathbf{S}_1^{-1/2},\qquad \mathbf{B}_r=\mathbf{W}_r^T\mathbf{B}=\mathbf{S}_1^{-1/2}\mathbf{Z}_1^T\left(\mathbf{L}^T\mathbf{B}\right),$$
$$\mathbf{C}_r=\mathbf{C}\mathbf{V}_r=\left(\mathbf{C}\mathbf{U}\right)\mathbf{Y}_1\mathbf{S}_1^{-1/2},\qquad \mathbf{M}_r=\mathbf{V}_r^T\mathbf{M}\mathbf{V}_r=\mathbf{S}_1^{-1/2}\mathbf{Y}_1^T\left(\mathbf{U}^T\mathbf{M}\mathbf{U}\right)\mathbf{Y}_1\mathbf{S}_1^{-1/2}.$$

3 Quadrature approximation of Gramians

- The BT method for LQOs, requires computation of five quantities, i.e.,

$$\mathbf{L}^\top\mathbf{U},\ \mathbf{L}^\top\mathbf{A}\mathbf{U},\ \mathbf{L}^\top\mathbf{B},\ \mathbf{C}\mathbf{U},\ \text{and}\ \mathbf{U}^\top\mathbf{M}\mathbf{U}.$$

- Approximate them with data-based quantities, like Quad-BT for purely linear systems [4].
- Replace the Gramians $\mathbf{P}$, $\mathbf{Q}_1$, and $\mathbf{Q}_2$ <u>implicitly</u> with quadrature-based Gramians:

$$\widetilde{\mathcal{Q}}=\widetilde{\mathbf{L}}\widetilde{\mathbf{L}}^\top=\sum_{j=1}^{N_q}\left(\phi_j e^{\mathbf{A}^\top\tau_j}\mathbf{C}^\top\right)\left(\phi_j e^{\mathbf{A}^\top\tau_j}\mathbf{C}^\top\right)^\top+\sum_{j=1}^{N_q}\left(\phi_j e^{\mathbf{A}^\top\tau_j}\mathbf{M}^\top\widetilde{\mathbf{U}}\right)\left(\phi_j e^{\mathbf{A}^\top\tau_j}\mathbf{M}^\top\widetilde{\mathbf{U}}\right)^\top$$

$$\widetilde{\mathbf{L}}=\begin{bmatrix}\widetilde{\mathbf{L}}_1 & \widetilde{\mathbf{L}}_2\end{bmatrix}=\begin{bmatrix}\phi_1 e^{\mathbf{A}^\top\tau_1}\mathbf{C}^\top & \cdots & \phi_{N_q}e^{\mathbf{A}^\top\tau_{N_q}}\mathbf{C}^\top \mid \phi_1 e^{\mathbf{A}^\top\tau_1}\mathbf{M}^\top\widetilde{\mathbf{U}} & \cdots & \phi_{N_q}e^{\mathbf{A}^\top\tau_{N_q}}\mathbf{M}^\top\widetilde{\mathbf{U}}\end{bmatrix}\in\mathbb{R}^{n\times(N_q+N_pN_q)} \tag{2}$$

$$\widetilde{\mathcal{P}}=\widetilde{\mathbf{U}}\widetilde{\mathbf{U}}^\top=\sum_{i=1}^{N_p}\left(\rho_i e^{\mathbf{A}t_i}\mathbf{B}\right)\left(\rho_i e^{\mathbf{A}t_i}\mathbf{B}\right)^\top,\quad \widetilde{\mathbf{U}}=\begin{bmatrix}\rho_1 e^{\mathbf{A}t_1}\mathbf{B} & \cdots & \rho_{N_p}e^{\mathbf{A}t_{N_p}}\mathbf{B}\end{bmatrix}\in\mathbb{R}^{n\times N_p}.$$

4 Replacing intrusive terms with data

We are approximating the matrices $\mathbf{L}^\top\mathbf{U}$, $\mathbf{L}^\top\mathbf{A}\mathbf{U}$, $\mathbf{L}^\top\mathbf{B}$, $\mathbf{C}\mathbf{U}$, and $\mathbf{U}^\top\mathbf{M}\mathbf{U}$ with surrogate quantities

$$\widetilde{\mathbb{H}}=\widetilde{\mathbf{L}}^\top\widetilde{\mathbf{U}},\ \widetilde{\mathbb{M}}=\widetilde{\mathbf{L}}^\top\mathbf{A}\widetilde{\mathbf{U}},\ \widetilde{\mathbb{h}}=\widetilde{\mathbf{L}}^\top\mathbf{B},\ \widetilde{\mathbb{g}}=\mathbf{C}\widetilde{\mathbf{U}},\ \text{and}\ \widetilde{\mathbb{N}}=\widetilde{\mathbf{U}}^\top\mathbf{M}\widetilde{\mathbf{U}}.$$

Theorem 1 ([5]). *For $\widetilde{\mathbf{U}}$ and $\widetilde{\mathbf{L}}$ as in (2), we have that matrices $\widetilde{\mathbb{H}}$, can be explicitly expressed as:*

$$\widetilde{\mathbb{H}}=\begin{bmatrix}\widetilde{\mathbb{H}}_1\\ \widetilde{\mathbb{H}}_2\end{bmatrix}=\begin{bmatrix}\widetilde{\mathbf{L}}_1^\top\widetilde{\mathbf{U}}\\ \widetilde{\mathbf{L}}_2^\top\widetilde{\mathbf{U}}\end{bmatrix}=\begin{cases}\rho_i\phi_j\mathbf{h}_1\left(t_i+\tau_j\right), & \text{for } \widetilde{\mathbf{L}}_1^\top\widetilde{\mathbf{U}}\in\mathbb{R}^{N_q\times N_p},\\ \rho_i\phi_j\rho_k\mathbf{h}_2(t_k,\tau_j+t_i), & \text{for } \widetilde{\mathbf{L}}_2^\top\widetilde{\mathbf{U}}\in\mathbb{R}^{N_qN_p\times N_p}.\end{cases} \tag{3}$$

where $1\le i,k\le N_p$, $1\le j\le N_q$. The other quadrature-based matrices can similarly be expressed in terms of samples of the time domain kernels and their derivatives.

5 Quadrature-based BT for LQO systems (QuadBT-LQO) [5]

- **Input**: Samples of the LQO kernels and their derivatives.
- Put together the data matrices: $\widetilde{\mathbb{H}}$, $\widetilde{\mathbb{N}}$, $\widetilde{\mathbb{M}}$, $\widetilde{\mathbb{h}}$, $\widetilde{\mathbb{g}}$.
- Compute the truncated **SVD** of matrix $\widetilde{\mathbb{H}}$:

$$\widetilde{\mathbb{H}}=\begin{bmatrix}\widetilde{\mathbf{Z}}_1 & \widetilde{\mathbf{Z}}_2\end{bmatrix}\begin{bmatrix}\widetilde{\mathbf{S}}_1 & \\ & \widetilde{\mathbf{S}}_2\end{bmatrix}\begin{bmatrix}\widetilde{\mathbf{Y}}_1^\top\\ \widetilde{\mathbf{Y}}_2^\top\end{bmatrix}.$$

- The matrices of the reduced-order data-driven system are given by,

$$\tilde{\mathbf{A}}_r=\cancel{\mathbf{W}_r^T\mathbf{A}\mathbf{V}_r}=\widetilde{\mathbf{S}}_1^{-1/2}\widetilde{\mathbf{Z}}_1^\top\left(\widetilde{\mathbb{M}}\right)\widetilde{\mathbf{Y}}_1\widetilde{\mathbf{S}}_1^{-1/2},\qquad \tilde{\mathbf{B}}_r=\cancel{\mathbf{W}_r^T\mathbf{B}}=\widetilde{\mathbf{S}}_1^{-1/2}\widetilde{\mathbf{Z}}_1^\top\left(\widetilde{\mathbb{g}}\right),$$
$$\tilde{\mathbf{C}}_r=\cancel{\mathbf{C}\mathbf{V}_r}=\left(\widetilde{\mathbb{h}}\right)\widetilde{\mathbf{Y}}_1\widetilde{\mathbf{S}}_1^{-1/2},\qquad \tilde{\mathbf{M}}_r=\cancel{\mathbf{V}_r^\top\mathbf{M}\mathbf{V}_r}=\widetilde{\mathbf{S}}_1^{-1/2}\widetilde{\mathbf{Y}}_1^\top\left(\widetilde{\mathbb{N}}\right)\widetilde{\mathbf{Y}}_1\widetilde{\mathbf{S}}_1^{-1/2}.$$

6 Using QDEIM for node selection

- **Problem**: The dimensions of $\widetilde{\mathbb{H}}$ scale quadratically with number of quadrature points.
- **Solution:** Subsample from $\widetilde{\mathbb{H}}_1=\widetilde{\mathbf{L}}_1\widetilde{\mathbf{U}}\in\mathbb{R}^{N_p\times N_q}$ like DEIM induced CUR factorization in [6].
- QDEIM [3] interpolates the rows of $\widetilde{\mathbb{H}}_1$ and returns n_1 indices (similarly for the columns $\rightsquigarrow\widetilde{\mathbb{H}}_1^\top$).
- In the following example, we have $N_p=N_q=6$ and $n_1=n_2=2$.

$$\widetilde{\mathbb{H}}_1=\begin{pmatrix}\mathbf{h}_1(t_1+\tau_1) & \mathbf{h}_1(t_2+\tau_1) & \mathbf{h}_1(t_3+\tau_1) & \mathbf{h}_1(t_4+\tau_1) & \mathbf{h}_1(t_5+\tau_1) & \mathbf{h}_1(t_6+\tau_1)\\ \mathbf{h}_1(t_1+\tau_2) & \mathbf{h}_1(t_2+\tau_2) & \mathbf{h}_1(t_3+\tau_2) & \mathbf{h}_1(t_4+\tau_2) & \mathbf{h}_1(t_5+\tau_2) & \mathbf{h}_1(t_6+\tau_2)\\ \mathbf{h}_1(t_1+\tau_3) & \mathbf{h}_1(t_2+\tau_3) & \mathbf{h}_1(t_3+\tau_3) & \mathbf{h}_1(t_4+\tau_3) & \mathbf{h}_1(t_5+\tau_3) & \mathbf{h}_1(t_6+\tau_3)\\ \mathbf{h}_1(t_1+\tau_4) & \mathbf{h}_1(t_2+\tau_4) & \mathbf{h}_1(t_3+\tau_4) & \mathbf{h}_1(t_4+\tau_4) & \mathbf{h}_1(t_5+\tau_4) & \mathbf{h}_1(t_6+\tau_4)\\ \mathbf{h}_1(t_1+\tau_5) & \mathbf{h}_1(t_2+\tau_5) & \mathbf{h}_1(t_3+\tau_5) & \mathbf{h}_1(t_4+\tau_5) & \mathbf{h}_1(t_5+\tau_5) & \mathbf{h}_1(t_6+\tau_5)\\ \mathbf{h}_1(t_1+\tau_6) & \mathbf{h}_1(t_2+\tau_6) & \mathbf{h}_1(t_3+\tau_6) & \mathbf{h}_1(t_4+\tau_6) & \mathbf{h}_1(t_5+\tau_6) & \mathbf{h}_1(t_6+\tau_6)\end{pmatrix}\qquad S_L=[t_1\quad t_4]\qquad S_U=[\tau_3\quad \tau_5]$$

7 Forming the data matrices

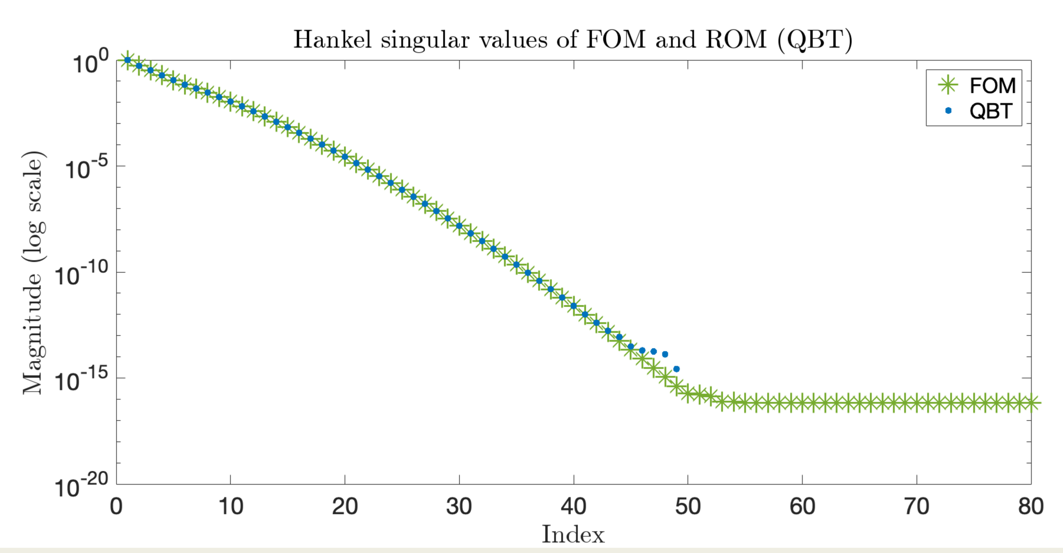
The indices selected by QDEIM are used to form the $\widetilde{\mathbb{H}}_2$ matrix. This provides significant reduction in the size of $\widetilde{\mathbb{H}}\in\mathbb{R}^{(N_q+n_1n_2)\times N_p}$.

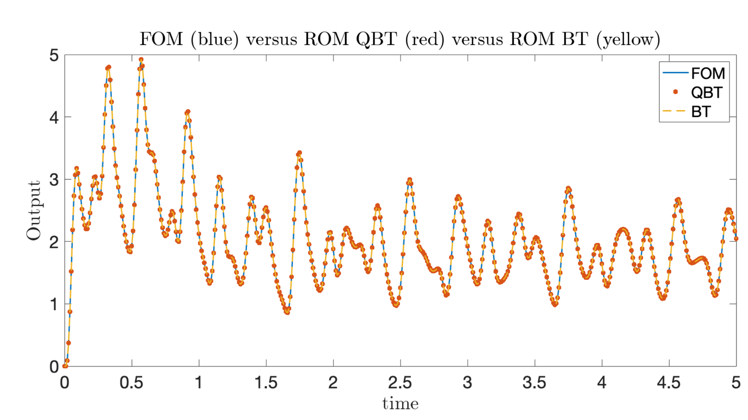
$$\widetilde{\mathbb{H}}_2=\begin{pmatrix}\mathbf{h}_2(t_1,t_1+\tau_3) & \mathbf{h}_2(t_1,t_2+\tau_3) & \mathbf{h}_2(t_1,t_3+\tau_3) & \mathbf{h}_2(t_1,t_4+\tau_3) & \mathbf{h}_2(t_1,t_5+\tau_3) & \mathbf{h}_2(t_1,t_6+\tau_3)\\ \mathbf{h}_2(t_1,t_1+\tau_5) & \mathbf{h}_2(t_1,t_2+\tau_5) & \mathbf{h}_2(t_1,t_3+\tau_5) & \mathbf{h}_2(t_1,t_4+\tau_5) & \mathbf{h}_2(t_1,t_5+\tau_5) & \mathbf{h}_2(t_1,t_6+\tau_5)\\ \mathbf{h}_2(t_4,t_1+\tau_3) & \mathbf{h}_2(t_4,t_2+\tau_3) & \mathbf{h}_2(t_4,t_3+\tau_3) & \mathbf{h}_2(t_4,t_4+\tau_3) & \mathbf{h}_2(t_4,t_5+\tau_3) & \mathbf{h}_2(t_4,t_6+\tau_3)\\ \mathbf{h}_2(t_4,t_1+\tau_5) & \mathbf{h}_2(t_4,t_2+\tau_5) & \mathbf{h}_2(t_4,t_3+\tau_5) & \mathbf{h}_2(t_4,t_4+\tau_5) & \mathbf{h}_2(t_4,t_5+\tau_5) & \mathbf{h}_2(t_4,t_6+\tau_5)\end{pmatrix}$$

- **Important**: We only form the data matrices $\widetilde{\mathbb{H}}$, $\widetilde{\mathbb{N}}$, $\widetilde{\mathbb{M}}$, $\widetilde{\mathbb{h}}$, $\widetilde{\mathbb{g}}$ and not $\widetilde{\mathbf{L}}$ or $\widetilde{\mathbf{U}}$.
- The idea of QDEIM is to subsample the quadrature nodes in a structure-preserving fashion.
- Effective computation speed-up.
- Numerical experiments suggest that QDEIM reliably identifies important time intervals in the system's impulse response, irrespective of the quadrature scheme being used.

8 Numerical results $\rightsquigarrow$ 1D advection-diffusion equation

- We use the model in [2] with a quadratic output term that appears in optimal control problems.
- Discretizing the PDE using finite difference schemes, results in a LQO system.
- We use the values $\alpha=1$, $\beta=0.01$ with 400 grid points $n=400$. Once again, we compute an ROM of order 49 using BT and QuadBT.
- We employ a standard trapezoid quadrature scheme with $N_p=N_q=800$, logarithmically spaced nodes in the interval $[10^{-4},10]$ for the QuadBT ROM.





$\lVert\mathbf{y}-\mathbf{y}_r\rVert_{\ell_\infty}$	2.8644e-13 (QBT)	2.6734e-13 (BT)
$\lVert\Sigma-\Sigma_r\rVert_{\mathcal{H}_2}$	1.3644 (QBT)	1.3644 (BT)

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